Solutions to Optics Chapter

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Problem (11.1): Deriving Snell's Law via Fermat's Principle

Problem Statement:
Fermat's Principle states that light takes the path that minimizes travel time. Apply this principle for two points on either side of a dielectric interface to derive Snell's Law.

Solution:

Step 1: **Travel Time in a Medium:** In a medium with refractive index n, the speed of light is $v = c/n$. For a path of length ℓ, the travel time is

$$T = \frac{n\ell}{c}.$$

(This follows from the definition of refractive index and the speed of light.)

Step 2: **Setting Up the Geometry:**
Consider a ray going from point A in medium 1 (index n_1) to point B in medium 2 (index n_2), crossing a flat interface at point P. Let $AP = \sqrt{x^2 + d_1^2}$ and $PB = \sqrt{(L-x)^2 + d_2^2}$, where d_1 and d_2 are the vertical distances from A and B to the interface, and L is the horizontal distance between A and B.

Step 3: **Total Time and Its Minimization:** The total travel time is

$$T(x) = \frac{n_1}{c}\sqrt{x^2 + d_1^2} + \frac{n_2}{c}\sqrt{(L-x)^2 + d_2^2}.$$

Fermat's Principle requires that the derivative $dT/dx = 0$. Carrying out the differentiation and setting it to zero leads (after some algebra) to

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where $\sin\theta_1 = \frac{x}{\sqrt{x^2+d_1^2}}$ and $\sin\theta_2 = \frac{L-x}{\sqrt{(L-x)^2+d_2^2}}$.

Step 4: **Conclusion:**
The condition

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

is Snell's Law. (This result corresponds to the boundary conditions expressed in Eqs. (10.7) and (10.9) of the document.)

Problem (11.2): Fresnel Equations, Reflectivity, Brewster and Critical Angles

Problem Statement:
(a) Use Fresnel's equations and the Poynting vectors to derive the reflectivity and transmissivity of a dielectric interface.
(b) For a glass–air interface (glass with $n = 1.5$, air with $n = 1$), compute the reflectivity at normal incidence.
(c) Determine the Brewster angle.
(d) Find the critical angle for total internal reflection.

Solution:

Step 1: **Fresnel Equations at Normal Incidence:** For normal incidence, the reflection coefficient (for the electric field) is

$$r = \frac{n_1 - n_2}{n_1 + n_2}.$$

(This follows from applying the boundary conditions [Eqs. (10.10)–(10.11)] at the interface.) The reflectivity (power reflection coefficient) is then

$$R = r^2.$$

Step 2: **Reflectivity for Glass-Air:**
Assuming light incident from glass ($n_1 = 1.5$) to air ($n_2 = 1$):

$$r = \frac{1.5 - 1}{1.5 + 1} = \frac{0.5}{2.5} = 0.2, \quad R = (0.2)^2 = 0.04.$$

So, 4% of the power is reflected.

Step 3: **Brewster's Angle:**
For parallel (p) polarization, the reflection goes to zero when

$$\tan\theta_B = \frac{n_2}{n_1}.$$

For light incident from air ($n_1 = 1$) onto glass ($n_2 = 1.5$):

$$\theta_B = \arctan(1.5) \approx 56.3^\circ.$$

Step 4: **Critical Angle:**
When light travels from a denser to a rarer medium (e.g., glass to air), the critical angle θ_c is given by

$$\sin\theta_c = \frac{n_2}{n_1} = \frac{1}{1.5} \approx 0.667, \quad \theta_c \approx 41.8^\circ.$$

Problem (11.3): Reflectivity of a Dielectric Layer

Problem Statement:
Consider a wave at normal incidence to a dielectric layer with index n_2 between media with indices n_1 and n_3.
(a) What is the reflectivity? (Think in terms of multiple reflections.)
(b) Find values for n_2 and the layer thickness d such that the reflection vanishes.

Solution:

Step 1: Multilayer Reflection:
When light is incident on a thin film (dielectric layer) of thickness d, the reflections from the two interfaces interfere. The effective reflection coefficient r_{eff} can be expressed as a sum of multiple reflections; the phase difference between successive reflections is

$$\phi = \frac{2\pi n_2 d}{\lambda}.$$

Step 2: Anti-Reflection Condition:
A common anti-reflection design sets

$$n_2 = \sqrt{n_1 n_3} \quad \text{and} \quad d = \frac{\lambda}{4n_2}.$$

Under these conditions, the reflections from the top and bottom surfaces cancel due to a 180° phase difference.

Step 3: Conclusion:
With $n_2 = \sqrt{n_1 n_3}$ and $d = \lambda/(4n_2)$, the overall reflection can vanish.

Problem (11.4): Ray Matrices and Image Plane of a Thin Lens

Problem Statement:
A ray starting at a height r_0 and slope r_0' is located a distance d_1 from a thin lens with focal length f. Use ray matrices to find the image plane where all rays starting at that point converge, and determine the magnification.

Solution:

Step 1: Ray Representation:
A ray is described by the vector

$$\begin{pmatrix} r \\ r' \end{pmatrix},$$

where r is the distance from the axis and $r' = \frac{dr}{dz}$ is the slope.

Step 2: Matrix for Free-Space Propagation:
Propagation over a distance d is given by:

$$M_{\text{free}} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}. \quad \text{(Eq. (10.37))}$$

Step 3: Matrix for a Thin Lens:
A thin lens with focal length f has the matrix:

$$M_{\text{lens}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}. \quad \text{(Eq. (10.39))}$$

Step 4: Combined System:
The overall system from the object plane (distance d_1) through the lens and then free space over distance d_2 is:

$$M = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix}.$$

The condition for convergence (i.e. all rays from the same object point meeting at the image) is that the output height becomes zero. Carrying out the multiplication leads to the standard thin lens (Gaussian) equation:

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}.$$

Step 5: Magnification:

The lateral magnification is given by:

$$M = -\frac{d_2}{d_1},$$

which tells us how the object height r_0 is scaled to form the image.

Conclusion:

Using ray matrices, we derive the thin lens equation and find that the image is formed at a distance d_2 such that $\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}$, with magnification $M = -d_2/d_1$.

Problem (11.5): Diffraction and Resolution for CD Players and Telescopes

Problem Statement:

A CD player uses an AlGaAs laser with a wavelength of 790 nm. (a) The pits on a CD have a diameter of roughly 1 µm; if the optics are diffraction-limited, what is the beam divergence angle? (b) Assuming the same geometry, what wavelength laser would be needed to read 0.1 µm pits? (c) How large must a telescope mirror be to read a car's license plate in visible light (wavelength 600 nm) from a Low Earth Orbit (200 km)?

Solution:

Step 1: Beam Divergence (Diffraction Limit):

For a circular aperture, the Rayleigh criterion gives the half-angle divergence as

$$\theta \approx 1.22\frac{\lambda}{D}.$$

If we assume that the effective aperture diameter D is about 1 µm (the size of the pits) and $\lambda = 790$ nm, then

$$\theta \approx 1.22\frac{790 \times 10^{-9}\,\mathrm{m}}{1 \times 10^{-6}\,\mathrm{m}} \approx 0.96\,\text{radians}.$$

(Note: This large angle reflects that a 1 µm aperture is very small relative to the wavelength.)

Step 2: Smaller Pits:

To read 0.1 µm pits, to maintain the same diffraction limit (i.e. the same relative spot size), the wavelength must scale down by a factor of 10. Thus, the required wavelength is

$$\lambda \approx \frac{790\,\mathrm{nm}}{10} \approx 79\,\mathrm{nm}.$$

This falls in the extreme ultraviolet region.

Step 3: Telescope Resolution for License Plate Reading:

The angular resolution of a circular aperture is

$$\theta \approx 1.22\frac{\lambda}{D}.$$

For a car's license plate of width roughly 0.3 m at a distance of 200 km, the required angular resolution is approximately

$$\theta \approx \frac{0.3\,\mathrm{m}}{200\,000\,\mathrm{m}} = 1.5 \times 10^{-6}\,\text{radians}.$$

Setting

$$1.22\frac{600 \times 10^{-9}}{D} = 1.5 \times 10^{-6},$$

solve for D:

$$D = 1.22\frac{600 \times 10^{-9}}{1.5 \times 10^{-6}} \approx \frac{732 \times 10^{-9}}{1.5 \times 10^{-6}} \approx 0.488\,\mathrm{m}.$$

Thus, a telescope mirror of roughly 0.5 m diameter is needed.

Conclusion:

For a CD with 1 µm pits, the diffraction-limited divergence is on the order of 1 radian. To resolve pits ten times smaller, a laser with approximately 79 nm wavelength is required. A 0.5 m telescope mirror is needed to resolve a license plate from 200 km using 600 nm light.

Problem (11.6): Dielectric Function from the Periodically Forced Lorentz Model

Problem Statement:

Solve the periodically forced Lorentz model for the dielectric constant as a function of frequency, and plot the real and imaginary parts.

Solution:

Step 1: Lorentz Oscillator Equation:

The motion of an electron under an oscillating field is given by

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = -eEe^{-i\omega t},$$

where m is the electron mass, γ is the damping rate, ω_0 is the resonant frequency, and E is the amplitude of the electric field. (This is analogous to Eq. (10.67) in the document.)

Step 2: Assumed Solution:

Assume a solution of the form $x = x_0 e^{-i\omega t}$. Then,

$$(-m\omega^2 - im\gamma\omega + m\omega_0^2)x_0 = -eE.$$

Solving for x_0:

$$x_0 = \frac{eE}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}.$$

Step 3: Relate to Polarization:

The polarization P is given by $P = Nex$, where N is the density of oscillators:

$$P = Nex_0 e^{-i\omega t} = \frac{Ne^2}{m(\omega_0^2 - \omega^2 - i\gamma\omega)} E e^{-i\omega t}.$$

Step 4: Dielectric Function:

The displacement field is $D = \epsilon_0 E + P$, and by definition $D = \epsilon(\omega) E$. Thus,

$$\epsilon(\omega) = \epsilon_0 + \frac{Ne^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}.$$

It is common to write the relative dielectric function as

$$\epsilon_r(\omega) = 1 + \frac{Ne^2}{\epsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}. \quad \text{(see discussion around Eq. (10.70))}$$

Step 5: Interpretation:

The real part of $\epsilon_r(\omega)$ gives the dispersion (variation of refractive index with frequency), and the imaginary part represents absorption (losses due to damping). A plot of these parts versus frequency would show a resonant behavior near ω_0, with the imaginary part peaking at resonance.

Conclusion:

The dielectric constant as a function of frequency is

$$\boxed{\epsilon_r(\omega) = 1 + \frac{Ne^2}{\epsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}.}$$

This result is derived from the forced Lorentz oscillator model (see Eqs. (10.67)–(10.70)) and forms the basis for understanding dispersion and absorption in dielectric materials.